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Numerical Results for the Estimation of Source Distributions from External Radiation-Field Measurements[1]

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ABSTRACT

The authors consider a transport process in a slab bounded by two parallel planes. Radiation is both absorbed and scattered isotropically. The radiation is due to continuously distributed internal isotropic sources. The aim is to estimate the distribution of the internal sources based on experimental measurements of the angular dependence of the emergent radiation.

First, a system of differential-integral equations for the emergent radiation is deduced. These are approximated by a system of ordinary differential equations, and some numerical results are given. Quasi-linearization is then used to solve the inverse problem numerically. The results of some computational experiments are given which indicate the sensitivity of the estimates of the source distribution to the observational errors in the emergent-radiation measurements. The methods can be generalized to the cases of anisotropic scattering and shell geometry.

INTRODUCTION

In an earlier paper [1] we outlined a method for determining the distribution of sources within a slab, given measurements of the emergent radiation. The purpose of this paper is to present the results of some numerical experiments. First we indicate the method, which takes full advantage of the modern computer's ability to integrate large systems of ordinary differential equations. This is followed by a presentation of some typical results. Errors in the observations produce errors in the estimation of the source distribution. Some indication of the seriousness of this problem is included. Lastly, a related approach to the inverse problem is described, and some numerical results are presented. The lines of approach sketched appear quite promising.

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1. Basic Equation for Emergent Intensity

Consider a homogeneous slab with horizontal plane parallel surfaces separated by a distance x_0. The slab absorbs radiation and scatters it isotropically, with albedo for single scattering λ. Isotropically emitting sources of radiation are contained in the slab. The strength of these sources $B(y)$ depends only on y, $0 \leqslant y \leqslant x_0$, the altitude above the bottom surface. The sources may be viewed as being arranged in layers.

Let

$$t(v, x) = \text{the intensity of radiation escaping from the upper surface of a slab of thickness } x \text{ in a direction whose direction cosine with respect to the upward-directed vertical is } v. \tag{1}$$

In an earlier note [1], we obtained the equation for $t(v, x)$ by an imbedding approach, in which the thickness x is varied by the addition of thin layers at the top. Let $X = X(v, x)$ and $Y = Y(v, x)$ be the X and Y functions of Chandrasekhar [2], respectively. The integro-differential equation for t is

$$t_x = \frac{1}{v} \left\{ -t + B(x)\, X(v, x) + \frac{\lambda}{2} X(v, x) \int_0^1 t(v', x)\, dv' \right\}, \tag{2}$$
$$t(v, 0) = 0.$$

The first term on the right represents an absorptive loss due to the addition of an elementary layer at the top, the second term represents emission and subsequent emergence of additional radiation, and the third term represents the interaction of radiation at the top, with subsequent emergence.

The X and Y functions satisfy the integro-differential equations

$$X_x = \frac{\lambda}{2} Y(v, x) \int_0^1 Y(v', x) \frac{dv'}{v'}, \qquad X(v, 0) = 1. \tag{3}$$

$$Y_x = -\frac{1}{v} Y(v, x) + \frac{\lambda}{2} X(v, x) \int_0^1 Y(v', x) \frac{dv'}{v'}, \qquad Y(v, 0) = 1. \tag{4}$$

Equations (2)–(4) form a complete system of equations for the determination of emergent intensity.

To obtain the computational solution, we approximate the system of equations by a system of ordinary differential equations. For this purpose, we replace the integrals by sums. We approximate an integral of a function using a Gaussian quadrature formula of order N,

$$\int_0^1 g(v)\, dv \cong \sum_{i=1}^{N} g(v_i)\, w_i, \tag{5}$$

where w_i, $i = 1, 2, \ldots, N$, are the Christoffel numbers, and v_i, $i = 1, 2, \ldots, N$, are the abscissas for the evaluation of the integrand. These are tabulated in [3]. From previous experience [4], [5], we expect that $N = 7$ will yield high accuracy.

Let us introduce the function $X_i(x)$, $Y_i(x)$, and $t_i(x)$, $0 \leqslant x \leqslant x_0$, as solutions of the system

$$\dot{X}_i = \frac{\lambda}{2} Y_i \sum_{j=1}^{N} Y_j \frac{w_j}{v_j}, \qquad X_i(0) = 1. \tag{6}$$

$$\dot{Y}_i = -\frac{1}{v_i} Y_i + \frac{\lambda}{2} X_i \sum_{j=1}^{N} Y_j \frac{w_j}{v_j}, \qquad Y_i(0) = 1, \tag{7}$$

$$\dot{t}_i = \frac{1}{v_i} \left\{ -t_i + BX_i + \frac{\lambda}{2} X_i \sum_{j=1}^{N} t_j w_j \right\}, \qquad t_i(0) = 0, \tag{8}$$

$$i = 1, 2, \ldots, N.$$

The dot indicates differentiation with respect to x. The integration of this system with the complete set of initial conditions is a routine matter on a modern computer for $0 \leqslant x \leqslant x_0$. For a given source distribution $B(y)$, $0 \leqslant y \leqslant x_0$, we obtain the intensities of the emergent radiation at the top, $t(v_i, x_0)$ for $i = 1, 2, \ldots, N$.

2. Some Numerical Results for Emergent Intensity

We use the basic equations of the previous section to compute emergent intensities for a variety of cases. First, we perform a series of experiments for the purpose of checking the FORTRAN program. We use $N = 7$ for the Gaussian quadratures and take an integration step of length 0.005. We consider the case of constant source distribution with $B(y) = 1.0$ and albedo 1.0. We compute $t(v)$ for a slab of thickness 1.0. Sobolev [6] gives the formula relating t to X and Y functions and moments of X and Y,

$$t(v) = [X(v) - Y(v)] \left\{ 1 - \frac{\lambda}{2} \left[\int_0^1 X(v)\, dv - \int_0^1 Y(v)\, dv \right] \right\}^{-1}. \tag{9}$$

The integrals of Eq. (9) are approximated by sums, using an N-point Gaussian quadrature formula. The right-hand side of Eq. (9) is easily calculated. It is compared with the computed $t(v)$. The formula is satisfied to at least six significant figures. For a thickness of 0.2 and $v = 0.5$, we compute $t = 0.445552$. The value 0.445 is given by Horak [7]. Figure 1 is a plot of emergent intensity t as a function of angle of emergence, arc cosine v, for a slab of thickness 1.5, albedo 1.0, and $B \equiv 1.0$.

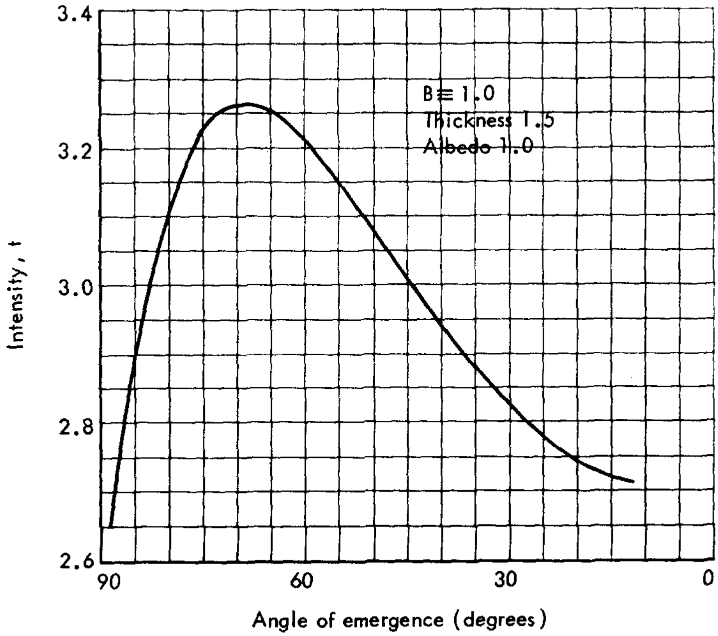

FIG. 1. Emergent intensity for a slab as a function of angle of emergence with $B = 1$.

Consider the case of parallel rays of radiation of net flux π per unit normal area incident, with direction cosine u, on the top of a slab which scatters light isotropically with albedo λ. We can compute the radiation which is diffusely reflected into the direction whose cosine is v, as reported in [8]. The emergent radiation for this case is the same as the radiation emerging from a slab possessing layers of emitting sources with a distribution

$$B(y) = \frac{\lambda}{4} \exp\left(-\frac{\text{thickness} - y}{u}\right). \tag{10}$$

We produce the emergent intensity both ways and compare the results. We find agreement to five to six significant figures, for a slab having albedo 1.0, thickness 10, and with incident direction cosine $u = 0.5$.

Next, we consider a series of slabs having thickness 1.5 and albedo 1.0. We call Slab 1 the slab which possesses a layer of sources of unit strength at the bottom, such that the source distribution function is

$$\begin{aligned} B(y) &= 1.0, \quad \text{for } 0 \leqslant y < \tfrac{1}{7}(1.5) \\ &= 0.0, \quad \text{for } \tfrac{1}{7}(1.5) \leqslant y \leqslant 1.5. \end{aligned} \tag{11}$$

Let Slab 2 refer to a slab with a layer of sources such that

$$\begin{aligned} B(y) &= 0.0, \quad \text{for } 0 \leqslant y < \tfrac{1}{7}(1.5) \\ &= 1.0, \quad \text{for } \tfrac{1}{7}(1.5) \leqslant y < \tfrac{2}{7}(1.5) \\ &= 0.0, \quad \text{for } \tfrac{2}{7}(1.5) \leqslant y \leqslant 1.5. \end{aligned} \tag{12}$$

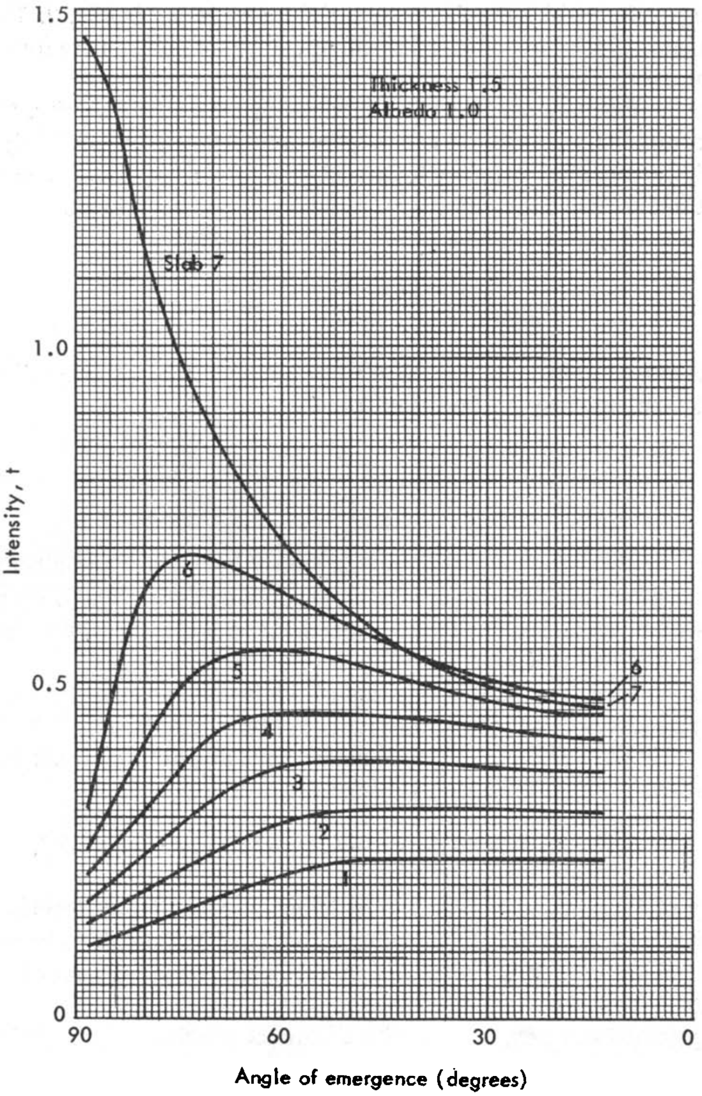

FIG. 2. Emergent intensities for slabs 1–7 as a function of angle of emergence.

Let Slabs 3–7 be defined in a similar fashion. Slab 7 then has

$$\begin{aligned} B(y) &= 0.0, \quad \text{for } 0 \leqslant y < \tfrac{6}{7}(1.5) \\ &= 1.0, \quad \text{for } \tfrac{6}{7}(1.5) \leqslant y \leqslant 1.5. \end{aligned} \tag{13}$$

The emergent intensities for these seven slabs are plotted in Fig. 2. The curve for Slab 7 in general lies above the other curves, since the sources are found closest to the top of the medium.

Due to the linearity of the basic equation for emergent intensity, this quantity may be computed with the aid of a superposition property. Let $B^{(i)}(y)$ be source distributions for K slabs, $i = 1, 2,..., K$, and let $t^{(i)}(v)$ be the corresponding emergent intensities. Then, if we have a slab whose distribution is

$$B(y) = \sum_{i=1}^{K} c_i B^{(i)}(y), \tag{14}$$

where the c's are constants, and if we write $t(v)$ as the corresponding emergent intensity, then $t(v)$ may be expressed as

$$t(v) = \sum_{i=1}^{K} c_i t^{(i)}(v). \tag{15}$$

Realizing that $t(v)$—for the case $B \equiv 1.0$, with thickness 1.5 and albedo 1.0—may be represented as the sum of the intensities for Slabs 1 through 7, we perform the necessary addition and find complete agreement with the intensity which is directly calculated, and which is plotted in Fig. 1.

Now let us turn to a slab of thickness 1.5 with albedo 1.0, and having a layer of emitting sources embedded within it. The layer is centered at altitude 0.55, and is of approximate thickness 0.3, with maximum strength about 0.9. We represent this situation by the distribution function

$$B(y) = 0.5[\tanh 10 \cdot (y - 0.4) - \tanh 10 \cdot (y - 0.7)], \tag{16}$$

for $0 \leqslant y \leqslant 1.5$, shown in Fig. 3. The intensity of radiation leaving the top of this medium in various directions is then shown in Fig. 4. As is to be expected, the curve resembles that for Slab 3 of Fig. 2, and is not at all like that of Fig. 1 which pertains to $B \equiv 1.0$.

Having gained some experience with the direct problem of calculating emergent intensities, we turn our attention to the inverse problem of estimating source distributions from knowledge of emergent intensities.

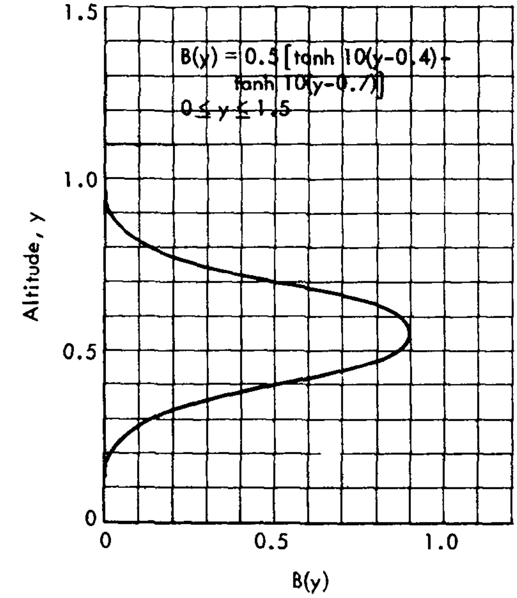

FIG. 3. Source distribution function.

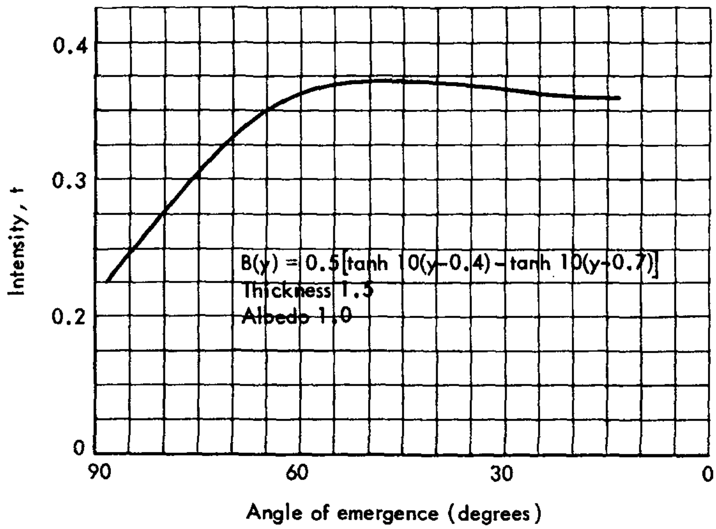

FIG. 4. Emergent intensity for a slab as a function of angle of emergence.

3. Estimation of Source Distributions

We pose the following inverse problem: Given the external intensity pattern for an emitting slab, determine the internal distribution of sources. For preciseness, consider the case in which the distribution function has the form

$$B(y) = 0.5[\tanh 10(y - a) - \tanh 10(y - b)], \tag{17}$$

where a and b are constants, and $0 \leqslant y \leqslant 1.5$. The thickness of the slab is known to be 1.5, and the albedo is 1.0. Suppose that the intensity has been measured in seven directions whose direction cosines are roots of the shifted Legendre polynomial of degree 7,

$$t_i(1.5) \cong b_i\,, \qquad i = 1, 2,..., 7. \tag{18}$$

The objective is to estimate $B(y)$ by determining the parameters a and b. Let the criterion be the minimization of the sum of the squares of the deviations between the theoretical solution $t_i(1.5)$ and the observations b_i, i.e., we wish to obtain

$$\min S, \tag{19}$$

where

$$S = \sum_{i=1}^{7} \{t_i(1.5) - b_i\}^2. \tag{20}$$

We use the method of quasilinearization described in [5]. This is discussed more fully in [9] and [10]. The X and Y functions which appear in the equations for $t(v)$ are found as solutions of a system of ordinary differential equations with known initial conditions. They are hence to be thought of as known functions of thickness. For the problem at hand, the basic differential equations are

$$\dot{t}_i = \frac{1}{v_i}\left\{-t_i + BX_i + \frac{\lambda}{2} X_i \sum_{j=1}^{7} t_j w_j\right\}, \qquad i = 1, 2,..., 7, \tag{21}$$

$$\dot{a} = 0, \tag{22}$$

$$\dot{b} = 0. \tag{23}$$

Equations (22) and (23) show that the constants a and b are to be dependent variables with zero derivatives. The independent variable is thickness x. The intensities satisfy known initial conditions,

$$t_i(0) = 0, \qquad i = 1, 2,..., 7. \tag{24}$$

The initial conditions for a and b are unknown, and it is precisely these initial values which we seek.

Let us suppose that we have the initial guesses a^0 and b^0. A numerical integration of the system of nonlinear Eqs. (21)–(23) with $a(0) = a^0$ and $b(0) = b^0$ generates the approximation of the intensities $t_i^0(x)$, $0 \leqslant x \leqslant 1.5$. The next approximation a^1, b^1, and $t_i^1(x)$ is found as the solution of a linear system of differential equations. These equations are obtained by expanding the right-hand sides of Eqs. (21)–(23) about the current approximation in a Taylor expansion and keeping only the linear terms.

We use the method of complementary solutions to effect the numerical solution. That is, we express the new approximation as a linear combination of complementary solutions and a particular solution. These solutions are generated by a numerical integration of the linear differential equations with a complete set of initial conditions. The multipliers of the complementary solutions are determined so as to minimize the sum S, which is, for the first approximation,

$$S = \sum_{i=1}^{7} \{t_i^1(1.5) - b_i\}^2. \tag{25}$$

They are found by solving a system of linear algebraic equations of order seven. These multipliers give us new estimates of a and b, as well as the complete next approximation $t_i^1(x)$ for $0 \leqslant x \leqslant 1.5$ and $i = 1, 2, \ldots, 7$.

The procedure is repeated to produce higher approximations. A sufficiently good initial approximation leads to quadratic convergence; that is, the number of correct digits in each estimate is doubled asymptotically. Practically speaking, this means that only a few iterations are required.

4. Some Numerical Results for the Estimation of Source Distributions

Some computational experiments are performed for the estimation of the source distribution. The true values of a and b in Eq. (17) are 0.4 and 0.7, respectively. The function B is shown in Fig. 3, and the function $t(v)$ in Fig. 4. Using seven accurate values of the emergent intensity, and starting with moderately poor initial guesses for the constants, we quickly converge to approximately the correct values. Next, we begin with correct values of a and b and go through three iterations of the method. We expect to retain the correct values as estimates. However, the estimates change slightly to

$$a \simeq 0.398 \qquad \text{and} \qquad b \simeq 0.699. \tag{26}$$

In the next three experiments, errors of varying amounts are introduced into the observations. Seven random numbers with a Gaussian distribution with 0.0

mean and standard deviation 1.0 are generated. These numbers are multiplied by 0.001 (0.1%), 0.01 (1%), and 0.02 (2%) times the correct intensities, respectively, in the three runs. The products are added to the accurate measurements to produce noisy measurements. Four iterations are carried out per run. The accuracies of the estimates of a and b are presented in Table I. As a spot check, the estimated strength of the sources at altitude 0.5 is compared with the correct value, and the error is given in the table. The last column of the table lists the sums of squares of deviations, S.

TABLE I

SOME NUMERICAL RESULTS

% Error in observations	% Error in a	% Error in b	% Error in $B(0.5)$	Sum S
0.0	−0.4	−0.14	+0.35	0.2×10^{-6}
0.1	−1.7	−0.69	+1.4	0.1×10^{-5}
1.0	−14.	−5.8	+6.4	0.8×10^{-4}
2.0	−29.5	−12.	+3.9	0.3×10^{-3}

From these results, it seems that the layer of sources may be shifted slightly up or down, with little effect on the emergent radiation. It is of interest to know the change in $t(v)$ due to a change in a and due to a change in b. Let

$$u_i(x) = \partial t_i(x)/\partial a, \tag{27}$$

$$z_i(x) = \partial t_i(x)/\partial b. \tag{28}$$

We perform partial differentiations on both sides of Eq. (8) and obtain

$$\dot{u}_i = \frac{1}{v_i}\left\{-u_i + B_a X_i + \frac{\lambda}{2} X_i \sum_{j=1}^{7} u_j w_j\right\}, \tag{29}$$

$$\dot{z}_i = \frac{1}{v_i}\left\{-z_i + B_b X_i + \frac{\lambda}{2} X_i \sum_{j=1}^{7} z_j w_j\right\}, \tag{30}$$

where

$$B_a = \frac{\partial B}{\partial a} = -5[1 - \tanh^2 10(x - 0.4)], \tag{31}$$

$$B_b = \frac{\partial B}{\partial b} = +5[1 - \tanh^2 10(x - 0.7)]. \tag{32}$$

For initial conditions, differentiation yields

$$u_i(0) = 0, \tag{33}$$

$$z_i(0) = 0. \tag{34}$$

It is apparent that $u_i(x)$ and $z_i(x)$ satisfy the same differential equations and initial conditions as $t_i(x)$; only the forcing functions are different. We compute u_i and z_i. The results are presented in Figs. 5 and 6. In Fig. 5, we show

$$u(v, x)|_{x=1.5} = \left.\frac{\partial t}{\partial b}\right|_{x=1.5}$$

and

$$-z(v, x)|_{x=1.5} = -\left.\frac{\partial t}{\partial b}\right|_{x=1.5},$$

plotted against direction cosine v. For comparison, Fig. 6, a replot of Fig. 4, is given. It appears that, if a is changed by a certain amount, b can be changed by a corresponding amount to keep the intensity pattern virtually the same.

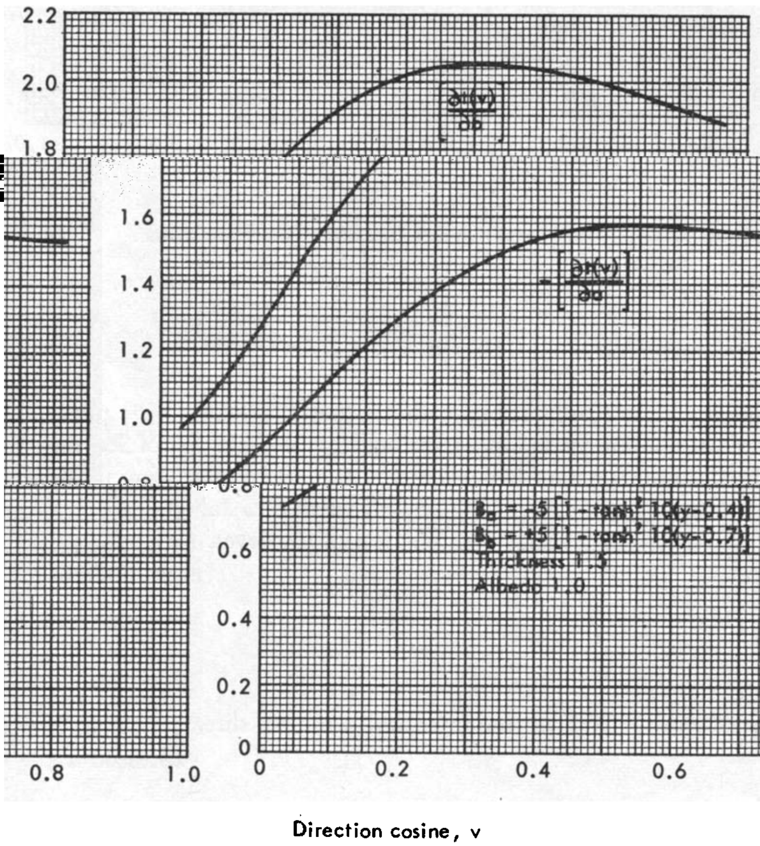

FIG. 5. Perturbation functions for a slab.

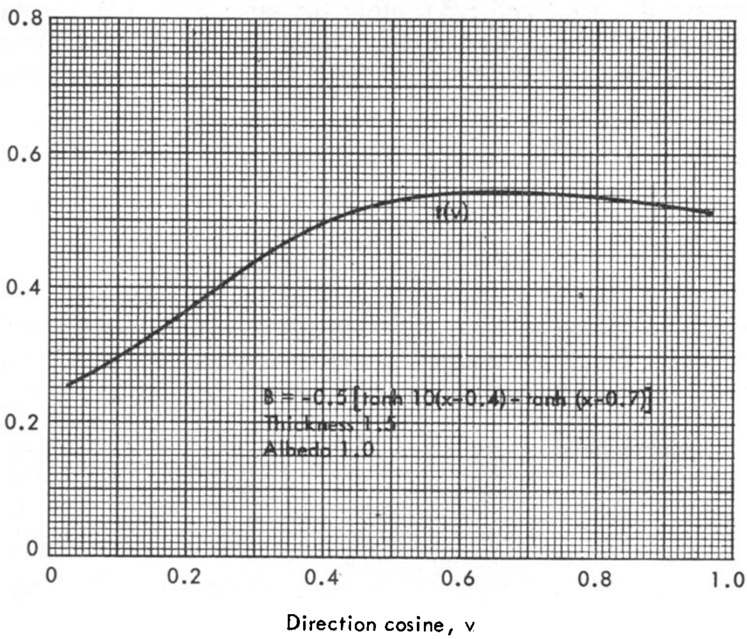

FIG. 6. Emergent intensity for a slab as a function of direction cosine.

5. Another Approach to the Estimation Problem

The superposition principle provides us with another approach to the problem of estimating source distributions. Let us again consider the Slabs 1–7 of Section II. Let the source distribution for the jth slab be $B^{(j)}$, and let the outgoing intensity in the direction whose cosine is v_i be $t_i^{(j)}$. Let us suppose that we have a slab whose source distribution β can be written as a linear combination of the seven sources in the seven different layers,

$$\beta = \sum_{k=1}^{7} \alpha_k B^{(k)}, \tag{35}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_7$ are seven constants. Then the intensity in the ith direction τ_i can be expressed as

$$\tau_i = \sum_{k=1}^{7} \alpha_k t_i^{(k)}, \qquad i = 1, 2, \ldots, 7. \tag{36}$$

Conversely, knowing the emergent intensity pattern for the entire slab and the pattern that results from a unit source in each layer, we may solve the linear algebraic equations for the multipliers $\alpha_1, \alpha_2, \ldots, \alpha_7$. This provides us with estimates of the strength in each layer.

We perform some computational experiments to see if we can determine the multipliers. First we let τ_i be the accurately known emergent intensity in the ith direction, for $i = 1, 2, \ldots, 7$, for a slab which is identical to Slab 1. Then, as a result of solving system (36), we find that

$$\alpha_1 = 1, \qquad \alpha_2 = \alpha_3 = \cdots = \alpha_7 = 0,$$

correct to eight significant figures. We perform similar experiments for slabs which are identical to Slabs 2–7, respectively, and we obtain the correct estimates of the multipliers.

Next, we introduce errors into the values of the intensities $\{\tau_i\}$. We repeat the experiments with random errors of 1% and 0.1%. The experiments fail, yielding constants which are completely erroneous. An examination of the inverse of the matrix whose elements are $\{t_i^{(k)}\}$ shows large positive and negative numbers. Thus, [redacted] This point will be discussed in subsequent papers.

6. Discussion

The foregoing results show that extremely high accuracies in the knowledge of external intensities in seven directions are required for satisfactory estimations of internal source distributions. (Of course, use of other criteria, such as the minimax criterion, could change this evaluation. In addition, various special techniques for solving ill-conditioned linear systems are available [11].) This fact suggests that increasing the number of observations may help to improve the estimates. The discussions of preceding sections have been limited to the special emergent directions arc cosine v_i, $i = 1, 2, \ldots, N$, where the cosines are roots of shifted Legendre polynomials. The purpose of producing $t(v, x)$ for these cosines is to approximate the integral $\int_0^1 t(v, x)\, dv$ with high precision. To compute the intensity $t(u, x)$ for an arbitrary direction cosine u, we add to the system of differential Eqs. (6)–(8) the following equations:

$$\dot{X}(u, x) = \frac{\lambda}{2} Y(u, x) \sum_{j=1}^{N} Y_j \frac{w_j}{v_j}, \qquad X(u, 0) = 1, \tag{38}$$

$$\dot{Y}(u, x) = -\frac{1}{u} Y(u, x) + \frac{\lambda}{2} X(u, x) \sum_{j=1}^{N} Y_j \frac{w_j}{v_j}, \qquad Y(u, 0) = 1, \tag{39}$$

$$\dot{t}(u, x) = \frac{1}{u} \left\{ -t(u, x) + BX(u, x) + \frac{\lambda}{2} X(u, x) \sum_{j=1}^{N} t_j w_j \right\},$$

$$t(u, 0) = 0, \tag{40}$$

and proceed as before.

Additional observations of t for arbitrary directions may be included in the inverse problem. For each new cosine u, three more differential equations are to be added to the system (21)–(23). This in turn results in more linear differential equations to be integrated in each approximation. Yet, the effort may be valuable in producing an improved estimation of the structure of the sources.

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